

FINITE GROUPS ADMITTING AN AUTOMORPHISM OF PRIME ORDER FIXING A CYCLIC 2-GROUP

TAKASHI OKUYAMA

(Received November 19, 1979)

1. Introduction

In this paper, we shall give a proof of the following Theorem, which is a conjecture of B. Rickman [9]; in special case, $C_c(\phi)$ has order 2, M.J. Collins and B. Rickman proved in [2].

Theorem. *Let G be a finite group which admits an automorphism ϕ of odd prime order r whose fixed-point-subgroup $C_c(\phi)$ is a cyclic 2-group. Then G is solvable.*

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [7].

An important tool that is brought to attack the problem is B. Baumann's classification of finite simple groups whose Sylow 2-subgroups are maximal [1], and in analogy with Matsuyama [8] that used the results of [1], we shall prove that $H_G(S; 2) \neq 1$, where S is a ϕ -invariant Sylow 3-subgroup of G .

C.A. Rowley has obtained a proof of the theorem under the additional hypothesis that G does not involve S_4 , the symmetric group on 4 letters.

The Theorem is a contribution to the continuing problem of showing that finite groups which admit an automorphism ϕ of odd prime order such that $C_c(\phi)$ is a 2-group are solvable.

2. Preliminaries

We first quote some frequently used results.

2.1. (Thompson [12])

Let G be a group which admits a fixed-point-free automorphism of prime order. Then G is nilpotent.

2.2. (Rowley [10])

Let G be a solvable group admitting an automorphism of odd prime order p such that $C_c(\phi)$, the fixed-point-subgroup of ϕ in G , is a cyclic q -group, $q \neq p$. Then, for any prime r , G is either r -nilpotent or r -closed.

2.3. (Glauberman [4])

Let G be a group with a Sylow p -subgroup P , either p odd or $p=2$ and S_4 is not involved in G , in which $C_G(Z(P))$ and $N_G(J(P))$ both have normal p -complements. Then G possesses a normal p -complement.

2.4. (Gilman and Gorenstein [3])

If G is a simple group with Sylow 2-subgroups of class 2, then $G \cong L_2(9)$, $q \equiv 7, 9 \pmod{16}$, A_7 , $Sz(2^n)$, n odd, $n > 1$, $U_3(2^n)$, $n \geq 2$, $L_3(2^n)$, $n \geq 2$, or $PSp(4, 2^n)$, $n \geq 2$.

2.5. (Gorenstein [7])

Let P be a Sylow p -subgroup of G , where p is the smallest prime in $\pi(G)$. If $p > 2$, assume $d_n(P) \leq 2$, while if $p=2$, assume P is cyclic. Then G has a normal P -complement.

2.6. (Matsuyama [8])

Let Q be a 2-group admitting an automorphism ϕ of odd order $\neq 1$. If $d_c(Q)=1$, then $Q=E*R$, where E is ϕ -invariant, extra-special or 1, and R is ϕ -invariant, and R is cyclic, D_m , Q_m , or S_m , $m \geq 4$.

2.7. (Collins-Rickman [2])

Let T be an extra-special 2-group admitting an automorphism ϕ of odd prime order r acting fixed-point-freely on T/T' . Let S be the natural semi-direct product $T \langle \phi \rangle$ and let K be a field of nonzero characteristic different from 2 and r . Assume that there exists a KS -module M for which $C_M(\phi) = C_M(T') = 0$.

Then (i) $r=2^n+1$ is a Fermat prime,

(ii) $|T|=2^{2n+1}$, and

(iii) $T \cong Q * \begin{pmatrix} n-1 & \\ & 1 \end{pmatrix} D$,

where Q and D denote the quaternion and dihedral groups of order 8, respectively, and $*$ denote the central product.

2.8. (Glauberman [5] [6])

Let G be a solvable group with a Sylow 2-subgroup Q with $G \neq C(Z(Q))N(J(Q))$, and $O(X)=1$. Put

$Z = \langle Z^* \mid G \triangleright Z^*: 2\text{-subgroup and } O_2(G/C(Z^*))=1 \rangle$

and $J = \langle x \in G \mid x: 2\text{-element, } |Z/C_Z(x)|=2 \rangle$

and $H = \langle J, C(Z) \rangle$. Then the following hold;

(i) there exists a normal subgroup G_i of H containing $C(Z)$, $1 \leq i \leq m$, such that, for $i=1, \dots, m$, $G_i/C(Z) \cong S_3$, and $H/C(Z) = G_1/C(Z) \times \dots \times G_m/C(Z)$.

(ii) let $V_i = [G_i, Z]$, $1 \leq i \leq m$, and let $V = V_1 \oplus \dots \oplus V_m$, then $Z = V \oplus C_Z(H)$ and $V_i \cong Z_2 \times Z_2$, $1 \leq i \leq m$.

(iii) there is a 3-element x_0 of H such that, for each $g \in H$, $H = \langle Q \cap H, x_0^g, C(Z) \rangle$ and $G/C(Z) = H/C(Z) C_{G/C(Z)}(x_0^g C(Z))$.

2.9. (Matsuyama [8])

Let G be a group with a Hall π -subgroup H , and let $1 \neq P \in \text{Syl}_3(H)$, Q 2-group. If $N_G(H) = HQ$, $d_e(Q) = 1$, $\Omega_1(Z(Q)) = \langle w \rangle$, $C_H(w) = 1$, and $\mathcal{U}_G(P; \pi') = 1$, then, for each $P^g \neq P$, $g \in G$, $m(P \cap P^g) \leq 1$.

2.10. (Burnside's theorem [7])

If a Sylow p -subgroup of G lies in the center of its normalizer in G , then G has a normal p -complement.

2.11. (Burnside's theorem [7])

If P is a Sylow p -subgroup of G , then two normal subsets of P are conjugate in G if and only if they are conjugate in $N_G(P)$. In particular, two elements of $Z(P)$ are conjugate in G if and only if they are conjugate in $N_G(P)$.

2.12. (Smith-Tyrer [11])

Let G be a group with an Abelian Sylow p -subgroup P for some odd prime p . If $[N(P):C(P)] = 2$ and $P \cap N(P)'$ is noncyclic, then G is p -solvable.

2.13. (Thompson Transitivity theorem [7])

Let G be a group in which the normalizer of every nonidentity p -subgroup is p -constrained. Then if $A \in \text{SCN}_3(P)$, $C_G(A)$ permutes transitively under conjugation the set of all maximal A -invariant q -subgroups of G for any prime $q \neq p$.

2.14. (Collins-Rickman [2])

Let G be a group, and let p and q be distinct prime divisors of G . Assume that G has an Abelian Sylow p -subgroup P for which $m(P) \geq 3$ and that, whenever P_0 is a subgroup of P with $m(P/P_0) \leq 2$, $N_G(P_0)$ is p -constrained. Then $C_G(P)$ permutes the elements of $\mathcal{U}_G^*(P; q)$ transitively under conjugation.

2.15. (Frobenius theorem [7])

G is p -nilpotent if and only if $N_G(H)/C_G(H)$ is a p -group for every nonidentity p -subgroup H of G .

3. The proof of the Theorem

Let G be a minimal counterexample to the Theorem, for the remainder of this paper.

Lemma 3.1. *G is simple.*

Proof. By Lemma 5.1. of [2].

Lemma 3.2. *Let p be a prime divisor of G and $P \in \text{Syl}_p(G)$. If $N_G(P)$ has a normal p -complement, then $p=2$ and the symmetric group S_4 is involved in G .*

Proof. By Lemma 5.2. of [2], (2.2) and (3.1).

For the remainder of this paper, Q denotes the ϕ -invariant Sylow 2-subgroup of G , and let $C_G(\phi) = \langle x \rangle$ and $\Omega_1(C_G(\phi)) = \langle w \rangle$.

Then Q is a unique ϕ -invariant Sylow 2-subgroup, and let p be an odd prime in $\pi(G)$ and $P \in \text{Syl}_p(G)$, then, by (3.2), $N_G(P) \ni w$.

Lemma 3.3. $d_c(Q) \geq 2$.

Proof. If $d_c(Q) = 1$, by (2.6) and hypothesis $Q = E * R$ where E is ϕ -invariant, extra-special, and R is ϕ -invariant, cyclic. If $E = 1$, by (2.5) G is 2-nilpotent, contrary to (3.1). So $E \neq 1$. Since $cl(Q) = 2$, by (2.4) this is a contradiction.

Lemma 3.4. *Every ϕ -invariant proper subgroup of G is 2-nilpotent.*

Proof. Assume otherwise. Let M be a non-nilpotent maximal ϕ -invariant subgroup of G without a normal Sylow 2-subgroup. If $N(O_2(M))$ is 2-nilpotent, M is nilpotent, a contradiction. By (2.2), $N(O_2(M))$ is 2-closed. Hence $M = N(O_2(M))$, $O_2(M) = Q$, and $M = N_G(Q)$. Thus there is an odd prime p dividing the index $[N_G(Q) : C_G(Q)]$.

By (3.3), there is a characteristic subgroup C of Q such that $C \cong Z_2 \times \cdots \times Z_2$, C contains $\Omega_1(Z(Q))$, and $[C, \phi] = 1$. Let P_0 be a ϕ -invariant Sylow p -subgroup of $N_G(Q)$ and P be a ϕ -invariant Sylow p -subgroup containing P_0 .

We now claim that $[C, P_0] = 1$. We may assume that $w \in C$. $[w, P_0] \subseteq Q \cap P = 1$. Since P_0 centralizes $C/C_G(P_0)$, $[P_0, C] = 1$. Thus $C \subseteq N_G(P_0)$.

Let M_0 be a maximal ϕ -invariant subgroup containing $N_G(P_0)$. If M_0 is 2-closed, $M_0 = N_G(Q)$. Since $N_P(P_0) = P_0$, $P = P_0$. Let Q_0 be a ϕ -invariant Sylow 2-subgroup of $N_G(P)$. Then $[P, Q_0] \subseteq P \cap Q = 1$, so $N_G(P)$ is P -nilpotent, and by (3.2), $p=2$, a contradiction. Thus M_0 is 2-nilpotent. Hence $M_0 = N_G(P)$. Since $C \subseteq N_G(P)$, $1 \neq [C, \phi] \subseteq C_G(P)$.

Now put $Z_0 = [\Omega_1(Z(Q)), \phi]$. If $Z_0 = 1$, $P, Q \subseteq C_G(Z_0)$. When $C_G(Z_0)$ is 2-closed, $P \subseteq N_G(Q)$, and $[Q_0, P_0] \subseteq Q \cap P = 1$, a contradiction. Hence $C_G(Z_0)$ is 2-nilpotent. Therefore as $Q \subseteq N_G(P)$, $[Q, P_0] \subseteq Q \cap P = 1$, a contradiction. Thus we may assume that $Z_0 = 1$, hence that $\Omega_1(Z(Q)) = \langle w \rangle$.

Put $\bar{Q} = Q / \langle w \rangle$ and let C_1 be the inverse image of $Z(\bar{Q}) \cap \bar{C}$ in Q . As $[C_1, x] \subseteq \langle w \rangle$, $C_1 \subseteq N_G(\langle x \rangle)$. On the other hand, let $y \in C_1$. Then $[y, \phi] \in C_G(\langle x \rangle)$, since $(y^{-1}xy)^\phi = y^{-1}xy$. Put $C_0 = [C_1, \phi]$, so that $1 \neq C_0 \subseteq N_G(P)$, hence $C_G(P_0)$ contains P_0 and x .

Now let M_1 be a maximal ϕ -invariant subgroup of G containing $C_G(C_0)$. If M_1 is 2-closed, $M_1 = N_G(Q)$, and $[Q_0, P] = 1$, contradiction. Thus M_1 is 2-nilpotent,

i.e. $M_1 = N_G(P)$.

Put $\tilde{Q} = Q/\Phi(Q)$. $[x, P_0] \subseteq P \cap Q = 1$. Since P_0 centralizes $\tilde{Q}/C_{\tilde{Q}}(P_0)$, P_0 centralizes \tilde{Q} . Hence $[P_0, Q] = 1$, a contradiction. Hence the lemma is proved.

For the remainder of this paper, in analogy with Matsuyama [8], we shall prove the following result;

- (i) $3 \nmid |G|$;
- (ii) $|C_G(S)|$ is odd, where S is a ϕ -invariant Sylow 3-subgroup of G ;
- (iii) $n_G(S; 2) \neq 1$; and
- (iv) $m(S) \geq 4$.

On the other hand, in analogy with Collins-Rickman [2], we shall prove that $n_G(S; 2) = 1$. Hence this contradicts above.

For the remainder of this paper, we shall write down the results which can be similarly proved as [8].

$$(3.5) \quad C_G(w) \subseteq Q.$$

$$(3.6) \quad \text{If } p \text{ is an odd prime in } \pi(G) \text{ and } P \in \text{Syl}_p(G), \text{ then } P \text{ is Abelian.}$$

$$(3.7) \quad \text{If } p \text{ is an odd prime in } \pi(G) \text{ and } A \text{ is any } p\text{-subgroup of } G, \text{ then } \text{Aut}_G(A) = N_G(A)/C_G(A) \text{ is a 2-group.}$$

$$(3.8) \quad \text{If } \Omega_1(Z(Q)) \neq \langle w \rangle, \text{ then } N_G(T) \text{ is a 2-group for any nontrivial } \phi\text{-invariant 2-subgroup } T \text{ of } G.$$

Now put P be a ϕ -invariant Sylow p -subgroup of G for any odd prime p in $\pi(G)$. Let K_p be a normal 2-complement of $N_G(P)$ and $Q_p = Q \cap N_G(P)$. Then $N_G(P) = Q_p K_p$, $Q_p \leq Q$. Furthermore let $Q_p^* = C_{Q_p}(K_p)$, and then $Q_p^* = [Q_p^*, \phi]$, since $w \in Q_p^*$.

Hence, for any $s \in \pi(K_s)$, $K_p = K_s$, $Q_p = Q_s$, and $Q_p^* = Q_s^*$. In particular, K_p is a nilpotent Hall subgroup of G .

$$(3.9) \quad C_{Q_p}(P) = Q_p^*.$$

$$(3.10) \quad d_c(Q_p/Q_p^*) = 1.$$

Furthermore let $M_p = N_G(P)$ and $\bar{M}_p = M_p/Q_p^* K_p$. Then by (2.6) and hypothesis, $\bar{M}_p = \bar{E}_p^* \bar{E}_p$, where either $\bar{E}_p = 1$ or \bar{E}_p is ϕ -invariant, extra-special and \bar{R}_p is ϕ -invariant, cyclic.

On the other hand, by (3.4), $N_G(Q)$ is nilpotent, and then $N_G(Q) = Q$ by (3.5). Hence by (3.2), S_4 is involved in G , yields $3 \nmid |G|$. Furthermore let $S \in \text{Syl}_3(G)$, and then $m(S) \geq 3$.

Lemma 3.11. *Let p be an odd prime in $\pi(G)$. We can write $\bar{M}_p = \bar{E}_p^* \bar{R}_p$, where either $\bar{E}_p = 1$ or \bar{E}_p is ϕ -invariant, extra-special, and \bar{R}_p is ϕ -invariant,*

cyclic.

If $\bar{E}_p \neq 1$, then $r=2^n+1$ is a Fermat prime.

Proof. By (2.7), it is immediate that $C_{\Omega_1(P)}(\phi) = C_{\Omega_1(P)}(\bar{E}_p') = 0$. By (2.7), it suffices to prove that ϕ acts on \bar{E}_p/\bar{E}_p' fixed-point-freely. First we may assume that $|\bar{R}_p|=2$. Then, since we can suppose that ϕ centralizes an element of \bar{E}_p of order 4, it is not necessarily trivial.

Now suppose that there exists an element \bar{y} of \bar{E}_p of order 4 such that $[\bar{y}, \phi]=1$. As \bar{E}_p is extra-special, the conjugate class of \bar{y} is $\{\bar{y}, \bar{y}w\}$. Hence $[\bar{E}_p: C_{\bar{E}_p}(\bar{y})]=2$. Then ϕ acts on the set, $\bar{E}_p - C_{\bar{E}_p}(\bar{y})$, fixed-point-freely. It is impossible.

Lemma 3.12. *Let S be a ϕ -invariant Sylow 3-subgroup of G . If $[Q_3/Q_3^*, \phi]=1$, then S is a T.I.-set.*

Proof. If not, there exists an element g of G such that $S^g \neq S$ and $S^g \cap S \neq 1$. First we shall show that $C_{Q_3}(z) = Q_3^*$ for any $z \in S^*$. It is immediate that $C_{Q_3}(z) \cong Q_3^*$. If $C_{Q_3}(z) \cong Q_3^*$ for some $z \in S^*$, $w \in C_{Q_3}(z)$, by hypothesis. But this is impossible. Next we will prove that, for any $z \in S^*$, $C_G(z)$ is 3-nilpotent.

Now put $C_G(z) = C$ and let S_1 be a nontrivial subgroup of S . By (3.7), $\text{Aut}_C(S_1)$ is a 2-group. Put $\text{Aut}_C(S_1) \ni t \neq 1$. Then t is a 2-element. Furthermore there exists an element y of S_1 such that $y^t \neq y$, i.e. y and y^t are conjugate in $C_G(z)$. By (2.11), y and y^t are conjugate in $N_C(S)$. Thus we may assume that $t \in N_C(S)$, and $t \in Q_3$. Then $t \in C_{Q_3}(z) = Q_3^* = C_{Q_3}(S)$, a contradiction. Hence $C_G(z)$ is 3-nilpotent by (2.15), especially $C_G(z)$ is 3-constrained.

Furthermore put $3 \neq p \in \pi(K_3)$, and let P be a ϕ -invariant Sylow p -subgroup of G . $N_G(S) = N_G(P)$. Thus $C_G(z)$ is $\pi(K_3)$ -nilpotent.

Next put $1 \neq y \in S^g \cap S$, and let M be a $\pi(K_3)$ -complement of $C_G(y)$, and then we will prove that M is a 2-group.

S normalizes M and $(|S|, |M|)=1$. Now suppose that M is not a 2-group. There exists an odd prime q in $\pi(M)$ such that $q \notin \pi(K_3)$. Furthermore there exists a Sylow q -subgroup Q_1 of M normalized by S . Since $\text{Aut}_C(Q_1)$ is a 2-group, $S \subseteq C_G(Q_1)$, and hence $Q_1 \subseteq K_3$. It is impossible. Thus M is a 2-group.

On the other hand, it is easy to show that $M \cong Q_3^*$. Now suppose that $M = Q_3^*$. Then $C_G(y) = Q_3^* K_3$, and since $S, S^g \subseteq C_G(y)$, $S = S^g$, a contradiction. Hence $M \cong Q_3^*$, $C_G(S) \subseteq N_G(M)$.

Let \bar{M} be the intersection of all elements of $\mathcal{U}_G^*(S; 2)$. By (2.14), $\bar{M} \cong M$. On the other hand, as \bar{M} is ϕ -invariant, $S\bar{M}$ is ϕ -invariant. By (3.4), $S\bar{M}$ is 2-nilpotent. Thus $[\bar{M}, S] \subseteq \bar{M} \cap S = 1$. Hence $\bar{M} \subseteq C_G(S) = Q_3^*$, a contradiction. This completes the proof of Lemma 3.12.

Now if $\bar{E}_3 \neq 1$, $r=2^n+1$ is a Fermat prime by (3.11), where $r=|\phi|$.

On the other hand, when $\bar{E}_3=1$, by (3.12), S is a T.I.-set, where S is a ϕ -invariant Sylow 3-subgroup of G .

By B. Baumann [1], Q is not a maximal subgroup of G , and thus there exists a proper subgroup X of G containing Q such that Q is a maximal subgroup of X .

In analogy with Matsuyama [8], we can say the following.

X is a solvable $\{2, 3\}$ -subgroup with $O(X)=1$, and X satisfies the hypothesis of (2.8). Thus the structure of X is one of the following two type.

<Type I>

$X/O_2(X)$ is isomorphic to S_3 , the symmetric group on 4 letters.

$Z(O_2(X))$ contains $Z(Q)$ and $Z(O_2(X)) = [Z(O_2(X)), X] \oplus C_{Z(O_2(X))}(X)$, where $[Z(O_2(X)), X]$ is isomorphic to $Z_2 \times Z_2$.

<Type II>

X has a subgroup H containing $O_2(X)$ such that $[X:H]=2$. $H/O_2(X) = X_1/O_2(X) \times X_2/O_2(X)$, $X_i/O_2(X)$ is isomorphic to S_3 , $i=1, 2$. $Z(O_2(X))$ contains $Z(Q)$ and $Z(O_2(X)) = [Z(O_2(X)), X_1] \oplus [Z(O_2(X)), X_2] \oplus C_{Z(O_2(X))}(H)$, where $[Z(O_2(X)), X_i]$ is isomorphic to $Z_2 \times Z_2$, $i=1, 2$.

On the other hand, considering the structure of X , $Z(Q)$ is noncyclic, by (3.8), $Q_3^*=1$.

Now we will show that $\mathcal{U}_G(K_3; 2) \neq 1$. For the remainder of this paper, let S be a ϕ -invariant Sylow 3-subgroup of G .

Lemma 3.13. $\mathcal{U}_G(S; \pi(K_3)') = \mathcal{U}_G(S; 2)$.

Proof. It is easy that $\mathcal{U}_G(S; \pi(K_3)') \supseteq \mathcal{U}_G(S; 2)$. If there exists an element A of $\mathcal{U}_G(S; \pi(K_3)')$ that is not a 2-group, by [7; 6.2.2], S normalizes some Sylow p -subgroup S^* of A . As $\text{Aut}_G(S^*)$ is a 2-group, $[S, S^*]=1$. But it contradicts $C_G(S)=K_3$.

By (3.13), it suffices to prove that $\mathcal{U}_G(S; \pi(K_3)') \neq 1$.

Now we suppose that $\mathcal{U}_G(S; \pi(K_3)')=1$. By Matsuyama [8], we can say the following.

(3.14) If $S^g \neq S$, $g \in G$, then $m(S \cap S^g) \leq 1$.

(3.15) There exists a nontrivial proper subgroup Z_1 of $Z(Q)$ such that $3 \nmid |C_G(Z_1)|$ and $[Z(Q):Z_1]=2$.

Furthermore, in analogy with Matsuyama [8], we can show the next lemma.

Lemma 3.16. *There exists a nontrivial element a of $\Omega_1(Z(Q))$ such that $|a^{\langle \phi \rangle} \cap Z_1| > \frac{1}{2} |a^{\langle \phi \rangle}|$ or $\Omega_1(Z(Q))^* = \{a^{\langle \phi \rangle}\}$.*

Proof. Put $a_1 \in \Omega_1(Z(Q))^* - A_1$, $a_1 \neq w$. Let $A_1 = \{a_1^{\langle \phi \rangle}\}$. If there exists an element of $\Omega_1(Z(Q))^* - A_1$ that does not equal w , let a_2 denote this element. So let $A_2 = \{a_2^{\langle \phi \rangle}\}$, and then $A_1 \cap A_2 = \phi$. Inductively, if there exists an element of $\Omega_1(Z(Q))^* - \bigcup_{k=1}^{i-1} A_k$ that does not equal w , we let a_i denote this element. Then we can write the following,

$$\Omega_1(Z(Q))^* - \langle w \rangle = \bigcup_{i=1}^m A_i,$$

where $A_i \cap A_j = \phi$ if $i \neq j$, $1 \leq i, j \leq m$.

Now suppose that $m \geq 2$. Let $|\Omega_1(Z(Q))| = 2^n$, and as $[\Omega_1(Z(Q)) : \Omega_1(Z_1)] = 2$, $|\Omega_1(Z_1)| = 2^{n-1}$. If, any i , $1 \leq i \leq m$, $|a_i^{\langle \phi \rangle} \cap Z_1| \leq \frac{1}{2} |a_i^{\langle \phi \rangle}|$, then since $|a_i^{\langle \phi \rangle}| = r$ is odd, $|\bigcup_{i=1}^m (a_i^{\langle \phi \rangle} \cap Z_1)| \leq |\Omega_1(Z(Q)) - \Omega_1(Z_1)| - 2$.

But, on the other hand, $|\Omega_1(Z(Q)) - \Omega_1(Z_1)| = 2^{n-1}$, and $|\Omega_1(Z_1)^*| = 2^{n-1} - 1$. It is impossible. Hence $m = 1$. $\Omega_1(Z(Q))^* = \{a_1^{\langle \phi \rangle}\}$. This lemma is proved.

(3.17) a^{ϕ^i} normalizes some Sylow 3-subgroup of G , $0 \leq i \leq r-1$.

Now put $\Delta_i = (a^{\phi^i})^G \cap Q_3$, and then $\Delta_i \neq \phi$, and $\Delta_i^{\phi} = \Delta_{i+1}$, $0 \leq i \leq r-1$. Furthermore, as $Q_3^* = 1$, $Q_3 = E_3 * R_3$.

If $E_3 = 1$, then S is a T.I.-set, by (3.13). In analogy with the above argument, we can show that $\Delta_i \neq \phi$, $0 < i < r-1$.

But, in this time, w is an only involution in Q_3 . This is a contradiction.

Hence, for the remainder of this paper, we may assume that $E_3 \neq 1$, i.e. $r = 2^n + 1$ is a Fermat prime. Then (3.16) is reduced that there exists a nontrivial element a of $\Omega_1(Z(Q))$ such that $|a^{\langle \phi \rangle} \cap Z_1| > \frac{1}{2} |a^{\langle \phi \rangle}|$.

On the other hand, $m(S) \geq 4$.

(3.18) There exists an element b_i, b_j of Δ_i, Δ_j , respectively, $0 \leq i, j \leq r-1$, $i \neq j$, $[b_i, b_j] = 1$.

Next Δ_i is determined as the following.

Lemma 3.19. $\Delta_i = \{b_i, b_i w\}$, $0 \leq i \leq r-1$, $b_i \neq w$.

Proof. If $w \in \Delta_i$, then w centralizes some element of order 3, a contradiction. Thus $w \notin \Delta_i$.

For the remainder, we set $b = b_i$.

Suppose that $b, b^g \in \Delta_i$, $g \in G$, $b \neq b^g$. Then $b, b^g \in Q_3$. Since $S = C_S(w) \oplus C_S(bw)$, $\frac{1}{2}m(S) = m(C_S(b)) = m(C_S(bw)) \geq 2$.

Let S^* be a Sylow 3-subgroup of $C_G(b^g)$ containing $C_S(b^g)$. There exists an element h of $C_G(b^g)$ such that $(C_S(b))^{g^h} \subseteq S^*$. On the other hand, let S_0 be a

Sylow 3-subgroup of G containing S^* , and then $S=S_0$ as $C_S(b^g) \subseteq S \cap S_0$. Since $(C_S(b))^{g^h} \subseteq S \cap S^{g^h}$, $gh \in N_G(S)$. Since $b^g = b^{g^h}$, b and b^g are conjugate in $N_G(S)$. As $N_G(S) = Q_3 K_3$, b and b^g are conjugate in Q_3 . Hence $\Delta_i = \{b_i, b_i w\}$.

Now put $\Delta = \langle \Delta_i \mid 0 \leq i \leq r-1 \rangle$, and then, by (3.19), Δ is ϕ -invariant Abelian. Furthermore, as $[\Delta, \phi] \neq 1$, $[\Delta, \phi] K_3$ is nilpotent, and

$$1 \neq [\Delta, \phi] \subseteq C_{Q_3}(K_3) = Q_3^* = 1,$$

this is a contradiction. Hence $N_G(S; 2) \neq 1$.

On the other hand, we will prove the next lemma, and then, in analogy with Collins-Rickman [2], the proof of the main theorem is complete.

Lemma 3.20. *Let S_0 be a proper subgroup of S such that $m(S/S_0) \leq 2$. Then $N_G(S_0)$ is 3-solvable.*

Proof. First we shall consider the case $m(S_0) > 2$. In this case, we will show that $C_G(S_0)$ is 3-nilpotent. Put $C = C_G(S_0)$, and let S_1 be a nontrivial subgroup of S . If there exists a nontrivial element t of $\text{Aut}_C(S_1)$, t is a 2-element as $\text{Aut}_C(S_1)$ is a 2-group. Then there exists an element y of S_1 such that $y^t \neq y$. Thus y and y^t are conjugate in $C_G(S_0)$. By (2.11), y and y^t are conjugate in $N_G(S)$. Hence we may assume that $t \in Q_3 \cap C \cong Z_2 \times \cdots \times Z_2$. As $t \neq w$, $S = C(t) \oplus C_S(tw)$. Hence

$$\frac{1}{2}m(S) = m(C_S(t)) = m(C_S(tw)).$$

This is a contradiction. By (2.15), $C_G(S_0)$ is 3-nilpotent. $C_G(S_0)/S_0$ is 3-solvable. Hence $N_G(S_0)$ is 3-solvable.

Now we may assume that $m(S) = 4$ and $m(S_0) = 2$. In this case, similarly, if $C_{M_3}(S_0) = C(S)$, $M_3 = N_G(S)$, then by (2.10), $C_G(S_0)$ is 3-nilpotent. Hence, furthermore, we may assume that $C_{M_3}(S_0) \cong C(S)$.

If there exists an element x_0 of $C_{M_3}(S_0)$ such that $|x_0| = 4$, then $x_0^2 = w \in C_{M_3}(S_0)$, a contradiction.

If there exists a four-group $\langle x_1 \rangle \times \langle x_2 \rangle$ in $C_{M_3}(S_0)$, then $S = \langle C_S(x_1), C_S(x_2), C_S(x_1 x_2) \rangle$. On the other hand, S_0 is contained in $C_S(x_1)$, $C_S(x_2)$, and $C_S(x_1 x_2)$, and since

$$m(C_S(x_1)) = m(C_S(x_2)) = m(C_S(x_1 x_2)) = 2,$$

$C_S(x_1) = C_S(x_2) = C_S(x_1 x_2) = S_0$, a contradiction. Hence we can write the following;

$$C_{M_3}(S_0) = C(S) \langle t \rangle,$$

where $t^2 \in C(S)$ and $S = S_0 \oplus [S, t]$.

Put $\overline{C_G(S_0)} = C_G(S_0)/S_0$. Then $\bar{S} = \bar{S} \cap N_{\overline{C_G(S_0)}}(\bar{S})'$. By (2.12), $\overline{C_G(S_0)}$ is 3-solvable. Hence, in this case, $N_G(S_0)$ is 3-solvable. This lemma is complete.

Now we already proved that $H_G^*(S; 2) \neq 1$. Next we will show that there exists a ϕ -invariant element Q_1 of $H_G^*(S; 2)$. Suppose false. Since $H_G^*(S; 2)$ is ϕ -invariant, r divides $|H_G^*(S; 2)|$. On the other hand, by (2.14), the element of $H_G^*(S; 2)$ permuted by $C(S)$ transitively. This is a contradiction.

Let $N = SQ_1$. By (3.4), N is nilpotent. Hence

$$Q_1 \subseteq C_G(S) = C_G(K_3).$$

On the other hand, as $Q_3^* = 1$, $|C_G(S)|$ is odd. This is a contradiction. The main theorem is proved.

Acknowledgement.

The author thanks to Mr. Fukushima and Mr. Matsuyama for this theme and for their helpful advices.

Reference

- [1] R. Baumann: *Endliche nicht auflösbare Gruppen mit einer nilpotenten maximalen Untergruppen*, J. Algebra **38** (1976), 119–135.
- [2] M.J. Collins and B. Rickman: *Finite groups admitting an automorphism with two fixed points*, J. Algebra **49** (1977), 547–563.
- [3] R. Gilman and D. Gorenstein: *Finite groups with Sylow 2-subgroups of class two*, I, II, Trans. Amer. Math. Soc. **207** (1975), 1–101: 103–125.
- [4] G. Glauberman: *Subgroups of finite groups*, Bull. Amer. Math. Soc. **73** (1967), 1–12.
- [5] G. Glauberman: *Failure of factorization in p -solvable group (I)*, Quart. J. Math. Oxford (2) **24** (1973), 71–77.
- [6] G. Glauberman: *Failure of factorization in p -solvable group (II)*, Quart. J. Math. Oxford (3) **26** (1975), 257–261.
- [7] D. Gorenstein: *Finite groups*, Harper and Row, New York, 1968.
- [8] H. Matsuyama: *Finite groups admitting an automorphism*, unpublished.
- [9] B. Rickman: *Groups admitting an automorphism of prime order fixing a cyclic subgroup of prime power order*; Quart. J. Math. Oxford (2) **26** (1975), 47–59.
- [10] C.A. Rowley: *Automorphisms of prime order fixing a cyclic group of prime power order*, J. London. Math. Soc. (2) **15** (1977), 91–98.
- [11] S.D. Smith and A.P. Tyrer: *On finite groups with a certain Sylow normalizer*, II, J. Algebra **26** (1973), 366–367.
- [12] J.G. Thompson: *Finite groups with fixed-point-free automorphisms of prime order*; Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578–581.

Sayama Industrial High School
Sayama-shi, Fujimi 2–5–1
Saitama-ken 350–13
Japan